

# Frozen randomness at the individual utility level

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## Abstract

Micro-simulated demand models require turning choice probabilities into a definitive outcome. This is commonly done by employing standard Monte Carlo methods. However, this method is inconsistent with random utility theory at the individual level. Since many demand models are based on random utility theory, this can lead to counter intuitive results when comparing single realisations of two scenarios. We here show how this can be avoided by freezing randomness at the individual utility level for the multinomial and nested logit models. We also implement the methodology in the ActivitySim framework and demonstrate its merit on a large-scale road capacity scenario in Brisbane.

## 1. Introduction

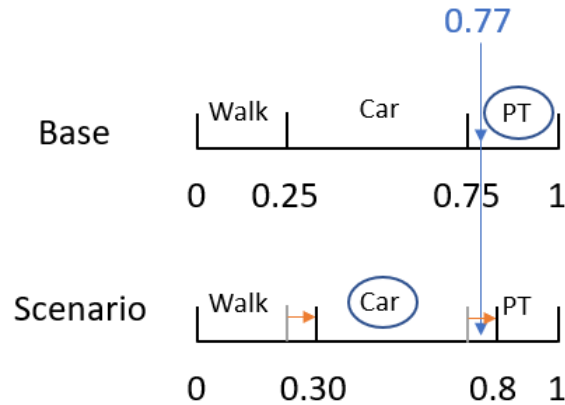
Modern econometric activity-based models (ABMs) require simulation during application: The calculation of all possible combinations of choice probabilities is computationally infeasible due to the sheer number of alternatives and instead at each model step a discrete choice is performed. In practice, the components of econometric ABMs are mostly multinomial logit (MNL) and nested logit (NL) models. These have closed-form solutions for the probability of choices and therefore drawing from these is often performed in a Monte Carlo way without drawing from the distribution of the unobserved part of utility directly, as described in Castiglione et al. (2015) and implemented in ActivitySim (2016). To remove simulation noise from results, the randomness of each model step is frozen such that if nothing changes, two runs will produce identical results. However, when comparing two scenarios, this methodology can lead to unexpected changes, as we will discuss in section 2. We elucidate the reason for this and propose a methodology to avoid this, based on standard multinomial and nested logit theory, in sections 2.1 and 2.2. In section 3, we will demonstrate the merits of our methodology on a real-world model scenario: We implement the methodology in ActivitySim and apply it to a model of South-East Queensland, comparing work location choice changes for a road capacity scenario for both the standard Monte Carlo ActivitySim choice method and our behaviourally more consistent method. Section 4 gives an outlook on future work before we conclude in section 5.

## 2. Methodology

We here motivate our methodology with a simple example before deriving it mathematically. Following the methodology in Castiglione et al. (2015) (Section 3.1.8.3), a possible way to turn a probabilistic choice into a definitive outcome is by performing a Monte Carlo draw from the probabilities of alternatives:

Write down the cumulative probability by ordering all alternatives in some prescribed way, then draw a standard uniform random number and pick the alternative associated with the cumulative probability at that number. We illustrate this in Fig. 1:

**Figure 1: Counter-intuitive choice switching example between a base and scenario case for a mode choice model. Improving the choice probability of the Walk alternative from 25% in the Base (upper plot) to 30% in the Scenario (lower plot) induces a switch from PT to Car for this particular draw of 0.77 of the random number (blue line), even though the attributes of those two alternatives did not change.**



The upper plot (Base) lines up three alternatives and their corresponding choice probabilities; Walk with probability 0.25, Car with probability 0.5, and PT with probability 0.25. Drawing a random uniform number, 0.77 here (blue number and arrow) determines the choice to be PT. Now imagine a scenario where the Walk alternative improves such that its choice probability is now 0.3; the other two alternatives decrease in probability accordingly. Freezing the random number generator for reproducibility means the choice maker in this example switches from PT to Car, as depicted in the lower plot (Scenario). This however is not expected because neither the attributes of the PT nor the Car alternative have changed and in a scenario comparison all variations not due to the introduction of the change in the scenario should be eliminated.

One way to avoid this switching is by utilising the econometric nature of the underlying models: Human behaviour is often modelled as a utility maximising process, see e.g. Ben-Akiva et al. (1985); Train (2009); Hensher et al. (2005): Given a choice between  $i$  mutually exclusive and exhaustive alternatives, a decision maker  $n$  will choose the one with maximum utility. In random utility theory, utility  $U_{ni}$  consists of two parts, an observable deterministic part  $V_{ni}$  and an unobservable random part  $\epsilon_{ni}$ . Assumptions on the distribution of  $\epsilon$  lead to different models. ActivitySim implements two kinds of discrete choice models, MNL and NL, see ActivitySim (2016). Instead of using the choice probabilities and drawing in the Monte Carlo way described earlier, we propose to draw the random error term directly. By fixing the random seed between scenarios, decision makers will always have the same unobserved preferences and will only change choices when the deterministic part of utility improves such that another alternative has maximum total utility. We describe how to achieve this in the following.

## 2.1 Logit model

For the multinomial logit model (MNL), the unobserved terms are assumed to be independent and identically distributed (i.i.d.) of type Gumbel. The Gumbel or  $EV1(\lambda, \mu)$  distribution with

location  $\lambda$  and scale  $\mu$  has cumulative distribution function (CDF)

$$F(x; \lambda, \mu) = \exp\left(-\exp\left(-\frac{x - \lambda}{\mu}\right)\right). \quad (1)$$

Drawing values from the underlying probability density function is then straight forward, see e.g. Train (2009): The inverse CDF is given by

$$F^{-1}(x) = \lambda - \mu \ln(-\ln(x)) \quad (2)$$

and drawing  $i$  times from a uniform distribution on  $(0, 1)$  and applying the inverse CDF leads to the desired distribution of error terms for an individual. Fixing the unobserved random part of utility between two scenarios can be achieved like for the Monte Carlo technique by ordering alternatives in a consistent way and freezing the random seed of the uniform distribution for each decision maker. Note that compared to the Monte Carlo method, which requires one random number per decision maker, we here need one random number per decision maker and alternative.

## 2.2 Nested logit

The NL model is a generalisation of the MNL model allowing for correlations of the unobserved term between alternatives within groups. This means the error terms are not independent and drawing from an inverse univariate CDF, as done for the MNL, is not possible anymore. However, following Ben-Akiva et al. (1985), the error term of the NL model can be decomposed into conditionally independent terms and drawing from the NL distribution becomes possible. Throughout this paper, we work with utility-maximising representation and RU2 scale normalisation, see e.g. chapter 7.4.3 of de Dios Ortuzar and Willumsen (2011).

### 2.2.1 Two levels

Following Ben-Akiva et al. (1985), we decompose the probability of choosing alternative  $j$  in nest  $k$  into the marginal probability of choosing nest  $k$  (marginalised over all alternatives  $j$  in  $k$ ) and the conditional probability of choosing  $j$  given that nest  $k$  was chosen. To be concrete, we here work with two alternatives in two nests. For a given realisation of the error term, the probabilities are indicators  $I[x]$ , where  $I[x] = 1$  if  $x$  is true and 0 otherwise. Nest 1 will be chosen if it contains the maximum utility, i.e.

$$I[\max_{j \in B_1} U_{1j} \geq \max_{i \in B_2} U_{2i}] = I[\max_{j \in B_1} V_{1j} + \varepsilon_{1j} \geq \max_{i \in B_2} V_{2i} + \varepsilon_{2i}]. \quad (3)$$

Ben-Akiva et al. (1985) conjectured that the error term could be written as

$$\varepsilon_{kj} = \tilde{\varepsilon}_k + \varepsilon_{j|k}, \quad (4)$$

where  $\varepsilon_{j|k}$  are i.i.d.  $EV(0, \mu_k)$ , and  $\tilde{\varepsilon}_k$  is independent of  $\varepsilon_{j|k}$ . The existence of this decomposition was proven to exist by Cardell (1997); Galichon (2022), and the authors showed that there is generally no finite closed-form expression for the probability density function of  $\tilde{\varepsilon}_k$ . We do not need to know its distribution here because it will drop out in the next steps. Using Eq. 4 inside the left-hand indicator of Eq. 3 leads to

$$\max_{j \in B_1} V_{1j} + \varepsilon_{1j} = \max_{j \in B_1} V_{1j} + \tilde{\varepsilon}_1 + \varepsilon_{j|1} = \tilde{\varepsilon}_1 + \max_{j \in B_1} V_{1j} + \varepsilon_{j|1}, \quad (5)$$

because  $\tilde{\epsilon}_1$  does not vary over alternatives in nest 1. Ben-Akiva et al. (1985) showed that

$$\max_{j \in B_1} V_{1j} + \epsilon_{j|1} \sim EV1(\mu_1 * \log \sum_{j \in B_1} \exp(V_j/\mu_1), \mu_1), \quad (6)$$

where we used  $\sim$  to denote "distributed as". Shifting all alternatives by the inclusive value, also called logsum,

$$IV_{B_1} = \log \sum_{j \in B_1} \exp(V_j/\mu_1), \quad (7)$$

leaves the maximum unchanged and using

$$\alpha * \epsilon + C \sim EV1(\alpha\lambda + C, \mu/\alpha) \text{ for } \alpha, C \in \mathbb{R}^+, \epsilon \sim EV1(\lambda, \mu) \quad (8)$$

in Eq. 5 we arrive at

$$\tilde{\epsilon}_1 + \max_{j \in B_1} V_{1j} + \epsilon_{j|1} = \tilde{\epsilon}_1 + \mu_1 IV_{B_1} + \tilde{\epsilon}_{j|1} = \mu_1 IV_{B_1} + \epsilon_{B_1}, \quad (9)$$

where

$$\epsilon_{B_1} \sim EV(0, 1) \quad (10)$$

by definition in Eq.(4). Performing the same calculation for nest 2 and plugging the results into Eq.(3) leads to

$$I[\max_{j \in B_1} U_{1j} \geq \max_{i \in B_2} U_{2j}] = I[\mu_1 IV_{B_1} + \epsilon_{B_1} \geq \mu_2 IV_{B_2} + \epsilon_{B_2}], \quad (11)$$

where  $\epsilon_{B_k} \sim EV(0, 1)$ . This equation is independent of  $\epsilon_{j|1}$ . We can therefore draw from  $\epsilon_{B_k}$  by using Eq.(2) and then choose the nest with maximum composite utility  $\mu_k IV_{B_k} + \epsilon_{B_k}$ .

The conditional choice within the chosen nest  $k = \textit{chosen}$  can then be determined by again using Eq.(2) to draw  $\epsilon_{j|chosen} \sim EV(0, \mu_{chosen})$ . This is due to  $\tilde{\epsilon}_k$  in Eq.(4) being constant in each nest and therefore removing it does not change the alternative with maximum utility in a given nest.

### 2.2.2 Three or more levels

The results from the previous section extend to more levels, see Ben-Akiva et al. (1985). We do not show this here, but note that the results can be read off existing works on closed-form probability solutions for NL models, see e.g. Newman et al. (2018): The probabilities of alternatives are given by products of independent marginal and conditional probabilities, which are all of MNL-type. Therefore, using the corresponding composite utility expressions, drawing from  $EV(0, 1)$  at each level and then further "walking down the tree" from the maximum composite alternative utility expression until one reaches a leaf (fundamental alternative) will lead to a sample from the NL probability density.

In summary, we have explicitly shown how to draw from both MNL and NL error terms directly and therefore how to make explicit choices in econometric ABMs consistent with random utility theory at the individual level for the MNL model, and nearly completely consistent for the NL model. Some edge cases still exist for the latter, due to using logsums. This could be avoided by not drawing conditionally, however this requires drawing from a distribution specified by its Laplace transform as described in Galichon (2022) and references therein and we have not explored this yet. We will return to this topic in section 4.

### 3. Practical Example

To illustrate the merits of our method in practice, we here briefly investigate a road capacity scenario with VLC’s ActivitySim model of South-East Queensland, see Zill et al. (2021) for details on the model and van Vuren et al. (2022) for more details on the scenario. In summary, we increase the capacity on one of inner Brisbane’s busiest roads, Coronation Drive, by two lanes in both directions. We perform four runs; a base and scenario run with the standard ActivitySim choice methodology and another base and scenario run with the methodology presented in section 2. To that end, we implemented this method in the ActivitySim software, making use of its open source nature by forking the project on github, see Zill (2022). In the following, we investigate the changes in workplace location choice across scenarios for each methodology. We chose the workplace location choice model because it is the first model for workers and the inputs to the runs are therefore the same for both methodologies; the skims differ of course between base and scenario.

Figure 2: Difference in workplace location choice between road capacity scenarios for ActivitySim’s standard choice method (Monte Carlo) and our method presented here (FRU) at different levels of spatial aggregation.

Choice method	Difference at SA4	Difference at SA3	Difference at SA2	Difference at TAZ
Monte Carlo	801	2037	6568	12657
FRU	721	1185	1586	1734
ratio FRU / MC	0.90	0.58	0.24	0.14

In figure 2, we show the total change in workplace locations at different levels of spatial aggregation as defined by the Australian Statistical Geography Standard, as well as at the travel analysis zone (TAZ) level. At the most aggregate level of SA4s, the difference in workplace locations between scenario and base is very similar for both the standard ActivitySim methodology (Monte Carlo) and our methodology (FRU). Looking into the spatial distribution (not shown here) confirms a very similar pattern. However, the more granular the spatial analysis unit becomes, the more the two methodologies start to differ: At the TAZ level, the Monte Carlo method leads to 12657 different workplace locations. In contrast, our method, which only leads to changes when the random utility of another location is larger in the scenario, shows 1734 changes. The large number of changes for the Monte Carlo method can be understood with the help of figure 1: Our model has 4380 TAZs and many of these will have small choice probabilities; correspondingly, their borders on the cumulative probability line will be very close. Any random number falling within that range will easily move to a different choice upon changes to the probability in any other zone, even when they are far away.

### 4. Current and future work

We saw in the previous section that freezing randomness at the individual utility level leads to more much more stable workplace location choices. This is the first choice model in our ActivitySim model, which has many more downstream choice models. Each of these models should be more stable by themselves due to our methodology. Additionally, they will also have the same initial "starting point" more often (i.e., the same choices in upstream models). We are currently investigating what the implications for overall model convergence are.

Another area of interest is economic benefit calculation. These can currently be calculated either by logsums and aggregation, i.e. accessibilities by population segments by mode and time period (hour here), or by the rule of half across choice dimensions like destinations, modes,

and time periods. Using logsums, i.e. expectation values, will remove simulation noise but our methodology could still be helpful by ensuring better convergence. For the rule of half, our methodology could be helpful by removing simulation noise.

Finally, Galichon (2022) showed how to write the NL error term as a sum of independent error terms without recourse to conditional drawing. In other words, the author showed how to draw from Ben-Akiva's "intermediate distribution", c.f. equation 4. This potentially opens the door to consistent benefit calculations at the individual level for micro-simulated models.

## 5. Conclusions

We showed a simple way of drawing from the unobserved utility of MNL and NL models directly. The motivation for this was shown to come from econometric, micro-simulated ABMs where the current standard methodology can lead to inconsistent results for scenarios at the individual level. We explained this with a simple toy mode choice example. We also showed its relevance in practice by implementing the methodology in the ActivitySim framework and applying it to a large-scale road capacity scenario in Brisbane. We investigated the changes in workplace location across scenarios for both the standard Monte Carlo random draw strategy and our random utility draw strategy. The results showed that both methodologies agreed at very aggregate spatial units, but at the granular zonal level our methodology was much more stable. We also summarised our current efforts and future work, which is centred around investigating the influence of the methodology on model convergence and economic benefit calculation. Last but not least, we hope this work will help with communicating model results to a non-technical audience, where seemingly illogical results can lead to distrust of models.

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## References

- ActivitySim. Activitysim github repository, 2016. URL <https://github.com/ActivitySim/activitysim>.
- Ben-Akiva, M. E., Lerman, S. R., and Lerman, S. R. *Discrete choice analysis: theory and application to travel demand*, volume 9. MIT press, 1985.
- Cardell, N. S. Variance components structures for the extreme-value and logistic distributions with application to models of heterogeneity. *Econometric Theory*, 13(2):185–213, 1997.
- Castiglione, J., Bradley, M., and Gliebe, J. *Activity-Based Travel Demand Models: A Primer*. The National Academies Press, Washington, DC, 2015. doi: <https://doi.org/10.17226/22357>.
- de Dios Ortuzar, J. and Willumsen, L. G. *Modelling transport*. John Wiley & Sons, 2011.
- Galichon, A. On the representation of the nested logit model. *Econometric Theory*, 38(2): 370–380, 2022. doi: [10.1017/S026646662000047X](https://doi.org/10.1017/S026646662000047X).
- Hensher, D. A., Rose, J. M., and Greene, W. H. *Applied choice analysis: a primer*. Cambridge University Press, 2005.

Newman, J. P., Lurkin, V., and Garrow, L. A. Computational methods for estimating multinomial, nested, and cross-nested logit models that account for semi-aggregate data. *Journal of Choice Modelling*, 26:28–40, 2018.

Train, K. E. *Discrete choice methods with simulation*. Cambridge university press, 2009.

van Vuren, T., Zill, J., and Veitch, T. Testing transport projects and policies with activity based models: does it make a difference? Transport Practitioners Meeting, 2022. URL <https://www.ptrc-training.co.uk/Portals/1/2022TPM/Presentations/TRANSPORT%20MODELLING%20-%20Methodology.zip>

Zill, J. Activitysim repository fork, 2022. URL <https://github.com/janzill/activitysim>.

Zill, J., Navidi, Z., and Inayathusein, A. Lessons from the development and application of activity-based models in australia. Transport Practitioners Meeting, 2021. URL <https://www.ptrc-training.co.uk/Portals/1/2021TPM/Papers%20and%20Presentations/Day%20Two/Session%20One/Room%20Three%20-%20Modelling%20-%20Methodology.zip>.