# Modelling and optimising fuel consumption in traffic assignment problems

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## Abstract

The Traffic Assignment (TA) Problem models route choices of users of a road transport network assuming a known relationship between traffic flow and travel time, and fixed demand between origin and destination points in the network. By modelling network user route choices, TA is able to derive network wide flows, for instance to understand the effects of modifications of the transport system in terms of congestion, travel times, or generalised cost. A basic assumption of TA is that network users selfishly minimise their own travel time and that a TA solution follows the so-called user equilibrium. It is well known that user equilibrium traffic flow does not necessarily follow a system-optimal travel pattern (in terms of travel time or generalised cost). Such a system-optimal travel pattern can be computed and congestion pricing theory shows that it can be enforced in a user equilibrium TA by charging network users an appropriate congestion toll (cf. Patrikkson 1994).

In this paper, we develop a model of fuel consumption within the TA framework. Our aim is to derive a system-optimal distribution of traffic with respect to fuel consumption and to devise congestion pricing and speed limits to encourage traffic flow to follow this system-optimal distribution when network users minimize a weighted sum of their travel time, fuel consumption, and tolls. We initially propose to apply a simplified model of fuel consumption (Song et al., 2013). Unfortunately, fuel consumption is not an increasing function of speed or arc flow – a basic assumption required to ensure that TA models converge to user equilibrium or system optimum solutions. Hence, considering fuel consumption in TA models provides new methodical challenges. Despite this, we analyse the proposed fuel-consumption TA model from a theoretical point of view and are able to show that, under appropriate assumptions, system-optimal traffic patterns can be derived and enforced by congestion pricing and appropriate speed limits. Moreover, we present results for standard TA instances (Transportation Networks for Research, 2016).

## 1. Introduction

Energy is a scarce resource and reduction of energy use is on the agenda of many governments worldwide. Energy use in transport is a large component of overall energy use. In New Zealand for instance, transport consumed 36% of the overall consumer energy in 2014 (MBIE, 2015, 2015b) and is currently 98% reliant on oil as transport fuel (Ministry of Transport, 2013). Internationally, 63.7% of the world oil consumption is in the transport sector (IEA, 2015). Our aim in the following is to gauge fuel consumption of car-based transport networks by developing strategic transport planning models, known as traffic assignment (TA) models, that capture fuel consumption based on traffic flow patterns. We use the developed models to identify traffic patterns with better overall fuel consumption given the same transport demand and we use them to derive network interventions such as speed limits and congestion pricing that can shift the transport system towards a fuel-optimal state.

TA models route choices of users of a road transport network assuming a known relationship between traffic flow and travel time, and fixed demand between origin and destination points in the network. It is usually assumed that network users aim to minimise their own travel time

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or a generalised cost function that captures time and other aspects of travel such as cost. It is assumed that *user equilibrium (UE)* network flows behave according to Wardrop's first principle (Wardrop, 1952) "The journey times on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route." It is well known that this leads to inefficiency of the transport system when compared to a *system-optimal (SO)* solution (according to Wardrop's second principle) where network flows are distributed to minimise overall travel time in the system. Congestion pricing theory has been applied to show that tolls can be used in order to enforce SO network flows while network users follow the UE principle with respect to a generalised cost function with toll added as an additional term. For TA theory and algorithms, we refer to Patriksson (1994); for congestion pricing (tolling), we refer to Lindsey and Verhoef (2001) or Fleischer et al. (2004). While many other types of TA models exist, such as TA with elastic demand, stochastic TA, or dynamic TA, we focus here on basic TA with known demand.

We aim to derive traffic patterns with system-optimal total fuel consumption. We investigate how tolling and the introduction of speed limits can be used to enforce system-optimal total fuel consumption in a network where route choice is governed by the UE principle with respect to a generalised cost function considering travel time, fuel consumption, and toll cost. Since fuel consumption is not an increasing function of arc flow, this generalised cost function will in general be non-monotonic in the amount of flow on an arc. Thus, the ususal assumption of increasing generalized cost functions for the users is not satisfied here, which leads to new methodical challenges.

In Section 2, we give a brief overview of the TA literature considering fuel consumption or (more commonly) emissions models and also the modelling of fuel consumption and emissions. In Section 3, our mathematical TA model is introduced. We show that flows with system-optimal total fuel consumption can be obtained by solving a nonlinear optimisation problem and that they can be enforced by tolling and setting appropriate speed limits. In Section 4, we present results for medium-sized transport networks.

# 2. Literature overview

## 2.1 Traffic assignment problems considering fuel consumption or emissions

In the literature, some efforts have been made to include fuel consumption or emissions in traffic assignment problems. Here we report on models capturing fuel consumption as well as on models capturing other emissions since functions relating speed and fuel consumption or speed and emissions follow a similar shape. Hence, their respective inclusion in TA models presents similar challenges. In general, both fuel consumption and emissions per km travelled are relatively high at low and high speeds, and lower in between (cf. Song et al., 2013). Hence, they do usually not satisfy the assumption of increasing functions common to TA problems, which are used in order to derive uniqueness of solutions. Most research in the past has focused on TA and emissions models rather than fuel consumption.

Tzeng and Chen (1993) are the first to propose a SO TA problem with a flow-dependent pollution function, but solve a problem in which pollution is of the form $p_0 + p_1 f_a$, where $p_0$ is a fixed pollution level and $p_1$ the additional flow-dependent pollution for arc flow $f_a$.

Rilett and Benedek (1994) and Benedek and Rilett (1998) discuss the inclusion of environmental factors and equity objectives in TA. Models proposed are macroscopic evaluating flow-dependent emissions where CO emissions are modelled as $(Ae^{Bv_a})/(Cv_a)$, where $A, B, C$ are constants and $v_a$ is average arc velocity. The authors point out that the emissions function they use is not increasing and, therefore, uniqueness of solutions is not guaranteed.

Nagurney et al. (2002) consider a UE TA model where a generalised cost function is derived based on the weighted sum of three objectives (travel time, cost, and pollution), all of which are assumed to be flow dependent and continuous. While existence of a solution is guaranteed due to the assumption that the three objectives are continuous, uniqueness is only shown for the very restrictive case of objectives with a special structure. The authors further consider a model with two objectives (cost (or time) and pollution) in the generalised cost function, where a flow-independent pollution rate is now assumed and pricing policies are derived.

Sugawara and Niemeier (2002) also model CO emissions. Again, an exponential function is used to fit speeds to emissions, a flow-emissions relationship is derived, and a SO TA formulation (with emissions objective) is stated. The authors note that the flow-emissions function is not increasing and propose to use an algorithm based on the Frank-Wolfe algorithm and a simulated annealing heuristic to identify a SO solution (with respect to emissions) despite problem non-convexity.

Yin and Lawphongpanich (2006) give an example showing that emissions for a UE TA solution can be lower than those for an SO solution (where route choice is with respect to travel time). CO emissions are modelled by applying an exponential function. The authors note that the emissions function is increasing as long as speed is not too high and assume an increasing emissions function throughout the paper. Assuming SO-fuel can be computed, the existence of tolling schemes is analysed.

Szeto et al. (2012) present a review of the literature on transport models including TA that capture some form of environmental sustainability, with a focus on models of emissions and noise, not necessarily fuel use.

Other research derives fuel consumption or emissions of a transport system after identifying TA travel patterns, in which case more sophisticated emissions models can be applied. For instance, Venigalla et al. (1999) introduce a TA model considering the portion of cold-starts on each arc as this most affects vehicle emissions and, thus, allows better modelling of emissions induced by a traffic pattern. Williams et al. (2001) also apply emissions rates to previously obtained TA travel patterns. However, these models can not be used to identify fuel-optimal or emissions-optimal travel patterns.

Some authors introduce bilevel formulations, where the top level problem seeks to optimise network-wide emissions and other network performance measures while, at the lower level, network users follow UE TA. A common example of this is network design problems, or second-best congestion pricing problems (cf Szeto et al., 2012).

## 2.2 Models for fuel consumption or emissions

Faris et al. (2011) review the literature on fuel consumption and emissions modelling. The conventional process of modelling fuel consumption or emissions in a transport network is outlined as follows: First, a transport model (such as TA) is run to obtain a network flow travel pattern, followed by the application of a fuel or emissions model. Fuel consumption and emissions models are categorised as microscopic, mesoscopic, and macroscopic models. Microscopic models take a vehicle's actual trajectory into account and carefully model, for example, acceleration and deceleration. Macroscopic models on the other hand are based on network-wide estimates and average arc parameters such as average speed. Mesoscopic models apply an intermediate level of detail.

In the context of TA, macroscopic models of fuel consumption and emissions are an appropriate choice since TA models are not usually able to derive network parameters at the level of detail required for meso- or microscopic models – at least when the aim is to include minimisation of fuel consumption or emissions in the TA model rather than to simply evaluate fuel consumption or emissions based on given TA solutions. Moreover, "macroscopic transport models are the most likely source of input data for emission

predictions" (Smit and Ntziachristos, 2012, p3). The research discussed in Section 2.1 also applies macroscopic models.

Several macroscopic emissions models have been used in the literature (cf. Rilett and Benedek, 1994; Yin and Lawphongpanich, 2006). In this paper, we apply a recent fuel consumption model developed by Song et al. (2013), where fuel consumption is modelled separately for light and heavy vehicles as shown in Figure 2 below and has the form

$$\frac{a}{v_a} + \mathcal{b} - c \cdot v_a + d \cdot v_a^2,$$

where $v_a$ is average speed (km/h) and the parameters $a, \mathcal{b}, c, d$ are calibrated for fuel consumption and other types of emissions in the paper.

A limitation of our work is that, by choosing a macroscopic model of fuel consumption depending only on average speed, our results will not reflect increased fuel consumption due to acceleration and deceleration. Smit et al. (2008) also comment on the fact that fuel consumption at a certain average speed may vary depending on whether the traffic situation is free flow (e.g., a speed of 30 km/h if this is the speed limit), or stop-and-go (e.g., a speed of 30 km/h on a congested motorway). Another limitation is that road gradient affects fuel consumption (Boriboonsomsin and Barth, 2009), which is also omitted from our model. We plan to include these aspects as we further develop our models in the future.

# 3. Traffic assignment with fuel consumption objective

## 3.1. The traffic assignment model

We now present our TA model with the objective of minimising fuel consumption.

We are given a road transport network represented by a directed graph $G = (V, A)$ and $K$ *commodities (or origin-destination pairs)* $\{(s_i, t_i, d_i)\}_{i=1}^K$, where $s_i \in V$ and $t_i \in V$ denote the *source* and *sink node* of commodity $i$, respectively, and $d_i \in \mathbb{R}^+$ denotes the *demand* of commodity $i$. We let $\mathcal{P}_i$ denote the set of all (simple) paths in $G$ from $s_i$ to $t_i$ and write $\mathcal{P} \coloneqq \cup_{i=1}^K \mathcal{P}_i$. To avoid trivialities, we assume that $\mathcal{P}_i \neq \emptyset$ for all $i$. A *path flow* $F$ is represented by a vector of nonnegative values $(F_p^i)$ for every $i = 1, \dots, K$ and every $p \in \mathcal{P}_i$. A path flow $F$ is called *feasible* if $\sum_{p \in \mathcal{P}_i} F_p^i = d_i$ for all $i$. An *arc flow* is a vector $(f_a)_{a \in A} \in \mathbb{R}^A$. Every path flow $F$ induces an arc flow $f$ via $f_a \coloneqq \sum_i \sum_{p \in \mathcal{P}_i : a \in p} F_p^i$ and an arc flow $f$ is called *feasible* if there exists a feasible path flow $F$ that induces $f$.

The travel time $t_a(f_a)$ on an arc $a \in A$ is assumed to be a positive-valued, strictly increasing, continuous function of the flow $f_a$ on the arc. The value $t_a(0)$ represents the *free flow travel time* on arc $a$ without any speed limit (determined by the characteristics of the road) and will usually be abbreviated by $t_a^0$. The *free flow speed* $v_a^0$ on arc $a$ is then given as $v_a^0 = \frac{s_a}{t_a^0}$, where $s_a > 0$ denotes the length of arc $a$ in kilometres, and the speed on an arc $a \in A$ at arc flow $f_a$ is given as

$$v_a(f_a) = \frac{s_a}{t_a(f_a)}$$

If a speed limit $v_a^{max}$ (in km/h) is imposed on arc $a$, the speed on this arc changes to

$$\bar{v}_a(f_a) = \min\{v_a(f_a), v_a^{max}\}$$

and the travel time function changes to

$$\bar{t}_a(f_a) = \frac{s_a}{\bar{v}_a(f_a)} = \max\left\{\frac{s_a}{v_a^{max}}, t_a(f_a)\right\}.$$

While our theoretical results hold for all travel time functions with the properties mentioned above, the travel time functions in the TA instances used in our case studies in Section 4 are given by $t_a(f_a) = t_a^0 \cdot \left(1 + \alpha \cdot \left(\frac{f_a}{k_a}\right)^\beta\right)$ as proposed by the Bureau of Public Roads (1964) (BPR), where $k_a$ is the so-called *practical capacity* of the arc, and $\alpha, \beta > 0$ are positive parameters (where typical values are $\alpha = 0.15$ and $\beta = 4$). Note that, since $\alpha, \beta > 0$, these functions are strictly increasing as required.

Figure 1 shows both the original BPR function $t_a(f_a)$ and the derived flow-speed relationship $v_a(f_a)$ as well as their versions $\bar{t}_a(f_a)$ and $\bar{v}_a(f_a)$ with additional speed limit $v_a^{max}$.

**Figure 1: $t_a(f_a)$ and $v_a(f_a)$ (dotted) and $\bar{t}_a(f_a)$ and $\bar{v}_a(f_a)$ with speed limit (solid) for the BPR function.**
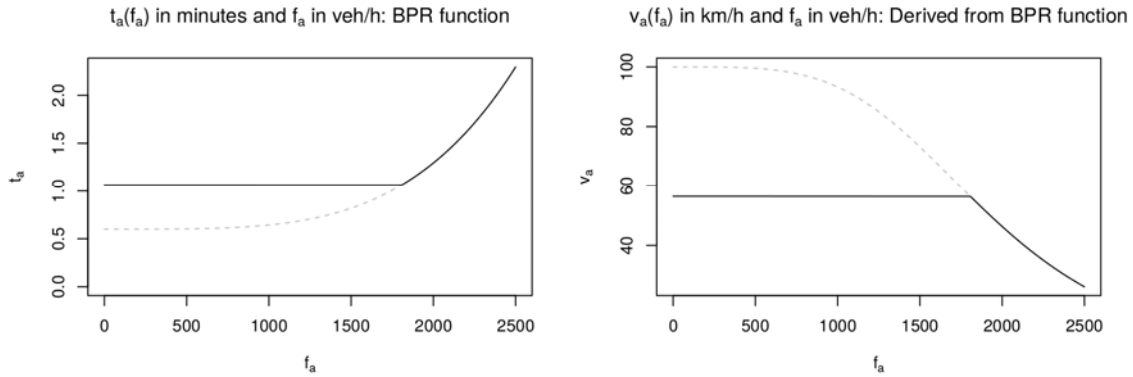


The speed-dependent fuel consumption $c_a(v_a)$ (in g/km) of each vehicle/user using an arc $a \in A$ is assumed to be of the form proposed in Song et al. (2013), that is

$$c_a(v_a) = \frac{a}{v_a} + \textit{b} - c \cdot v_a + d \cdot v_a^2 \text{ for } 0 < v_a \leq v_a^0,$$

where $a, \textit{b}, c, d$ are parameters with $a, d > 0$ that may depend on the arc. Observe that, under these assumptions, the fuel consumption $c_a(v_a)$ per vehicle on arc $a$ is a (continuous, nonnegative, and) *strictly convex* function of the speed $v_a$ on arc $a$. Hence, since $c_a(v_a) \to +\infty$ for $v_a \to 0$, there exists a unique fuel-optimal speed $v_a^{opt} \in (0, v_a^0]$ for each arc $a \in A$ such that $c_a(v_a)$ is strictly decreasing on $(0, v_a^{opt}]$ and strictly increasing on $[v_a^{opt}, v_a^0]$. Alternatively, we can express the fuel consumption of each user using an arc $a \in A$ as a function of the arc flow $f_a$ on $a$ by using the relation

$$c_a(f_a) = c_a(v_a(f_a)) \text{ or } \bar{c}_a(f_a) = c_a(\bar{v}_a(f_a))$$

depending on whether speed limits are imposed or not. Note that $v_a(f_a)$ and $\bar{v}_a(f_a)$ are decreasing in $f_a$. However, since $c_a(v_a)$ is *not* a monotone function of $v_a$ when considering the whole interval $(0, v_a^0]$, $c_a(f_a)$ and $\bar{c}_a(f_a)$ are *not* monotone in $f_a$ on $[0, +\infty)$ (see also Figure 2).

With a fuel consumption per kilometre given as above, the (total) fuel consumption in grams of each vehicle/user using an arc $a \in A$ is given as the product of fuel consumption per kilometre and arc length, i.e., as $c_a(v_a) \cdot s_a$ or $\bar{c}_a(v_a) \cdot s_a$ depending on whether speed limits are imposed or not. Equivalently, as function of arc flow, the fuel consumption in grams of each user/vehicle on arc $a$ is given as $c_a(f_a) \cdot s_a$ or $\bar{c}_a(f_a) \cdot s_a$.

**Figure 2: Fuel consumption based on Song et al. (2013) and BPR travel time functions for light (red) and heavy (blue) vehicles.[1]**
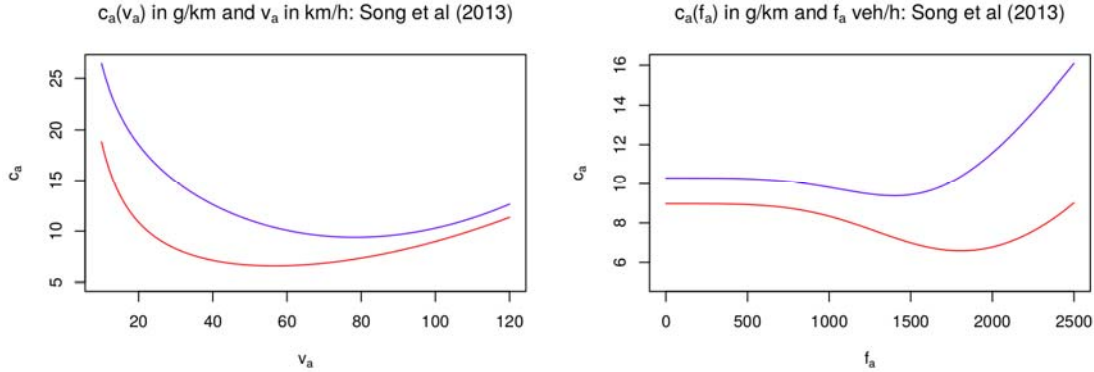


We now describe the objectives of the users in our TA model. As usual, we assume the flow of all commodities to consist of infinitesimally small users that behave selfishly. There are two non-negative sensitivity parameters $\gamma_t, \gamma_c \geq 0$ that measure the sensitivity of the users to travel time and fuel consumption, respectively (i.e., they convert travel time and fuel consumption into their monetary equivalent). Even though $\gamma_t, \gamma_c$ are allowed to be zero, we assume that *at least one* of the two sensitivity parameters is strictly positive (i.e., users are not completely insensitive to *both* travel time and fuel consumption). Given a vector $\tau = (\tau_a)_{a \in A}$ of nonnegative *tolls* on the arcs, each user of commodity $i$ chooses a path $p \in \mathcal{P}_i$ that minimizes the weighted sum of travel time, fuel consumption, and tolls the user experiences on this path. More precisely, for every path flow $F$ inducing arc flow $f$, the *generalised cost* experienced by each user using an arc $a \in A$ is given as

$$C_a(f_a) := \gamma_t \cdot \bar{t}_a(f_a) + \gamma_c \cdot \bar{c}_a(f_a) \cdot s_a + \tau_a$$

and the generalized cost of a path $p \in \mathcal{P}_i$ for the users of commodity $i$ is given as

$$C_p(F) := C_p(f) := \sum_{a \in p} C_a(f_a).$$

Note that we formulated the generalised costs of the users for the setting with speed limits on the arcs. The generalised costs for the setting without speed limits are obtained by replacing $\bar{t}$ by $t$ and $\bar{c}$ by $c$ above.

By Wardrop's first principle (Wardrop, 1952), a path flow $F$ is a user equilibrium if, for every $i \in \{i, \dots, K\}$ and $p_1, p_2 \in \mathcal{P}_i$ with $F_{p_1}^i > 0$, we have $C_{p_1}(F) \leq C_{p_2}(F)$. Note that, since both the travel time and the fuel consumption experienced by each user on an arc $a$ are continuous functions of the flow $f_a$ on the arc, the existence of a user equilibrium for every choice of speed limits and tolls follows by standard results from the theory of variational inequalities (cf. Nagurney et al. (2002)).

While the objective of each user is to minimize their generalised cost, our global objective is to choose vectors $v^{max} = (v_a^{max})_{a \in A}$ of speed limits and $\tau = (\tau_a)_{a \in A}$ of tolls on the arcs in order to minimize the total fuel consumption obtained in a user equilibrium flow $F$ resulting from these speed limits and tolls.

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[1] The function was multiplied by a factor of 10 to obtain the same range of g/km fuel use as in the corresponding figures in Song et al. (2013), and a more realistic fuel use in l/km.

## 3.2. A traffic pattern with system-optimal total fuel consumption

In this section, we formulate the optimisation problem of computing a traffic pattern with system-optimal total fuel consumption, show that it admits an optimal solution, and prove some important properties of optimal solutions.

In our setting, we consider a *(feasible) traffic pattern* $(F, v)$ to consist of a feasible path flow $F$ together with a vector $v = (v_a)_{a \in A}$ of speeds to be driven on the arcs of the traffic network satisfying the condition that $0 < v_a \leq v_a(f_a)$ for each arc $a \in A$, where $f$ denotes the arc flow induced by $F$. Thus, we demand that the chosen speed on each arc $a$ is positive and no higher than the speed $v_a(f_a)$ obtained on the arc when no speed limit is imposed and the amount of flow (traffic) on $a$ is $f_a$. The optimisation problem of determining a traffic pattern with system-optimal total fuel consumption then consists of determining a traffic pattern $(F, v)$ minimizing the total fuel consumption

$$c(F, v) := \sum_{a \in A} f_a \cdot c_a(v_a) \cdot s_a.$$

In other words, the optimisation problem consists of choosing the paths of all users and, for each arc $a \in A$, a speed $v_a$ at which the arc is traversed such that the total fuel consumption resulting from the chosen paths and speeds is minimized. Formally, this problem can be written as the following nonlinear optimization problem (NLP) in the variables $F_p^i, f_a, v_a$:

$$\min \sum_{a \in A} f_a \cdot c_a(v_a) \cdot s_a$$

$$s.t. \sum_{p \in \mathcal{P}_i} F_p^i = d_i, \qquad i = 1, \dots, K$$

$$f_a = \sum_{i} \sum_{p \in \mathcal{P}_i : a \in p} F_p^i, \qquad a \in A$$

$$F_p^i \geq 0, \qquad i = 1, \dots, , K, p \in \mathcal{P}_i$$

$$0 < v_a \leq v_a(f_a), \qquad a \in A.$$

Note that, even though the objective function is continuous, the existence of an optimal solution of this problem is not guaranteed a priori since the feasible region is not compact due to the strict inequality in the last constraint. The existence of an optimal solution will follow using the strict convexity of $c_a(v_a)$ and the following lemma:

**Lemma 1:**

Given a feasible flow $F$ inducing arc flow $f$, the vector $v^* = (v_a^*)_{a \in A}$ of speeds given by

$$v_a^* := \min\{v_a^{opt}, v_a(f_a)\} \text{ for all } a \in A$$

minimises the total fuel consumption.

**Proof:**

This follows since, by strict convexity, $c_a(v_a)$ (and, hence, also $c_a(v_a) \cdot s_a$ since $s_a > 0$ is a positive constant) is strictly decreasing on $(0, v_a^{opt}]$ and strictly increasing on $[v_a^{opt}, v_a^0]$. ∎

Using Lemma 1, the problem of computing a traffic pattern with system-optimal total fuel consumption simplifies to

$$\min \sum_{a \in A} f_a \cdot c_a \big( \min\{v_a^{opt}, v_a(f_a)\} \big) \cdot s_a$$

$$s.t. \sum_{p \in \mathcal{P}_i} F_p^i = d_i, \qquad i = 1, \dots, K$$

$$f_a = \sum_i \sum_{p \in \mathcal{P}_i : a \in p} F_p^i, \qquad a \in A$$

$$F_p^i \geq 0, \qquad i = 1, \dots, K, p \in \mathcal{P}_i.$$

Note that the objective function is still continuous and the feasible region is now compact, so an optimal solution exists. Together with Lemma 1, we have, thus, shown the following result:

**Theorem 2:**

There exists a traffic pattern $(F^*, v^*)$ with system-optimal total fuel consumption such that

$$v_a^* := \min\{v_a^{opt}, v_a(f_a^*)\} \text{ for all } a \in A,$$

where $f^*$ denotes the arc flow induced by $F^*$. ∎

Under the assumptions made in Sections 3.1 and 3.2, the simplified problem of computing a traffic pattern with system-optimal total fuel consumption with objective $\min \sum_{a \in A} f_a \cdot c_a \big( \min\{v_a^{opt}, v_a(f_a)\} \big) \cdot s_a$ is convex as long as the value $f_a^{opt}$ for which $v_a(f_a^{opt}) = v_a^{opt}$ is sufficiently large. This can be shown by exploiting the fact that a system-optimal flow can also be obtained by computing a user equilibrium with respect to the *marginal* arc cost functions $\frac{d}{df_a} f_a \cdot c_a \big( \min\{v_a^{opt}, v_a(f_a)\} \big) \cdot s_a$. As stated in Patriksson (1994), uniqueness of the system-optimal solution follows when these marginal arc cost functions are increasing. Under the assumptions made here, marginal arc cost functions are strictly increasing if $f_a^{opt} > \sqrt{2} \cdot k_a$ for parameters $\alpha = 0.15$ and $\beta = 4$ in the BPR function or if $f_a^{opt} > \sqrt[4]{\frac{3}{5}} \cdot k_a$ for parameters $\alpha = 1$ and $\beta = 4$ (details omitted). The two sets of parameters occur in the instances we use in Section 4.

## 3.3. Enforcing a fuel-optimal traffic pattern by speed limits and tolls

In this section, we prove the existence of speed limits $v^{max} = (v_a^{max})_{a \in A}$ and tolls $\tau = (\tau_a)_{a \in A}$ that enforce a fuel-optimal traffic pattern as a user equilibrium.

To this end, consider the situation that speed limits of $v_a^{max} := v_a^{opt}$ are imposed on all arcs $a \in A$. Since $v_a^* := \min\{v_a^{opt}, v_a(f_a^*)\} \leq v_a^{opt}$ for all $a \in A$, this choice of speed limits does not influence the feasibility of a fuel-optimal traffic pattern as in Theorem 2.

With these speed limits, $\bar{c}_a(f_a) = c_a\big(\bar{v}_a(f_a)\big)$ becomes an *increasing* function of the arc flow $f_a$. This follows since $\bar{v}_a(f_a) = \min\{v_a(f_a), v_a^{max}\}$ is decreasing in $f_a$ and $c_a(\cdot)$ is decreasing within the interval $(0, v_a^{opt}]$. Hence, since $\bar{v}_a(f_a) \in (0, v_a^{opt}]$ for all $f_a \geq 0$ when $v_a^{max} = v_a^{opt}$, we obtain that $\bar{c}_a(f_a) = c_a\big(\bar{v}_a(f_a)\big)$ is increasing in $f_a$. Consequently, since $\bar{t}_a(f_a) = \max\left\{\frac{s_a}{v_a^{max}}, t_a(f_a)\right\}$ is also increasing in $f_a$ and $s_a > 0$, $\gamma_t, \gamma_c \geq 0$, the generalised cost $C_a(f_a) = \gamma_t \cdot \bar{t}_a(f_a) + \gamma_c \cdot \bar{c}_a(f_a) \cdot s_a + \tau_a$ experienced by the users on an arc $a \in A$ also becomes an increasing function of the arc flow $f_a$.

Thus, given any nonnegative vector $\tau = (\tau_a)_{a \in A}$ of tolls and speed limits of $v_a^{max} = v_a^{opt}$ on all arcs $a \in A$, we are in the situation of continuous, (weakly) increasing generalised cost functions on the arcs, for which it is well-known that user equilibria are unique in the sense

that the generalised cost experienced by the users on each arc $a \in A$ is the same in every user equilibrium (cf. Beckmann et al. (1956), Dafermos and Sparrow (1969)). Moreover, since $v_a^{max} = v_a^{opt}$, both $\bar{t}_a(f_a)$ and $\bar{c}_a(f_a) \cdot s_a$ are constant up to the unique value $f_a^{opt}$ for which $v_a(f_a^{opt}) = v_a^{opt}$ and strictly increasing afterwards. Hence, the uniqueness of the generalised cost $C_a(f_a)$ on each arc $a$ in a user equilibrium also implies that both $\bar{t}_a(f_a)$ and $\bar{c}_a(f_a) \cdot s_a$ must be unique in every user equilibrium for every arc $a \in A$ (even if one of the sensitivity parameters $\gamma_t, \gamma_c$ is equal to zero). Hence, we obtain:

**Proposition 3:**

When speed limits of $v_a^{max} := v_a^{opt}$ are imposed on all arcs $a \in A$, the fuel consumption $\bar{c}_a(f_a) \cdot s_a$ and generalised cost $C_a(f_a)$ experienced by the users on each arc $a \in A$ become increasing functions of the arc flow $f_a$. Moreover, for every given vector $\tau = (\tau_a)_{a \in A}$ of tolls, the travel time $\bar{t}_a(f_a)$ and the fuel consumption $\bar{c}_a(f_a) \cdot s_a$ in a user equilibrium are uniquely determined for every arc $a \in A$. ∎

Note that, since all paths used by the users of a commodity in a user equilibrium must have the same generalised cost, Proposition 3 also implies that, given a vector $\tau = (\tau_a)_{a \in A}$ of tolls and speed limits of $v_a^{max} = v_a^{opt}$ on all arcs $a \in A$, the total generalised cost $\sum_{a \in A} f_a \cdot C_a(f_a)$ of all user equilibria must be the same. Thus, by the same arguments as above, also the total travel time $\sum_{a \in A} f_a \cdot \bar{t}_a(f_a)$ and the total fuel consumption $\sum_{a \in A} f_a \cdot \bar{c}_a(f_a) \cdot s_a$ of all user equilibria coincide.

Hence, we can now make the notion of *enforcing* a fuel-optimal traffic pattern precise: Given speed limits of $v_a^{max} = v_a^{opt}$ on all arcs $a \in A$, a vector $\tau = (\tau_a)_{a \in A}$ *enforces* a fuel-optimal traffic pattern if the total fuel consumption of all user equilibria for tolls $\tau$ equals the system-optimal total fuel consumption of a fuel-optimal traffic pattern as in Theorem 2.

Having established monotonicity of the fuel consumption and generalised cost on each arc $a \in A$ in the arc flow $f_a$ and essential uniqueness of user equilibria for every possible vector $\tau$ of tolls, the existence of tolls enforcing a fuel-optimal traffic pattern now follows by using a general result of Fleischer et al. (2004). Using Corollary 3.2 in Fleischer et al. (2004), we obtain the following result:

**Theorem 4:**

When speed limits of $v_a^{max} := v_a^{opt}$ are imposed on all arcs $a \in A$, there exists a vector $\tau = (\tau_a)_{a \in A}$ enforcing a fuel-optimal traffic pattern. ∎

Note that imposing the speed limits of $v_a^{max} := v_a^{opt}$ on all arcs is essential here since, as shown in Proposition 3, this ensures that the fuel consumption $\bar{c}_a(f_a)$ and generalised cost $C_a(f_a)$ experienced by the users on each arc $a \in A$ become *increasing* functions of the arc flow $f_a$, which is essential in order to be able to apply the result of Fleischer et al. (2004).

To summarise, a fuel-optimal traffic pattern can be computed based on the NLP in Section 3.2 immediately before Theorem 2. The speed limits can then be chosen as $v_a^{max} := v_a^{opt}$ for each arc $a \in A$ and the existence of tolls enforcing a fuel-optimal traffic pattern together with these speed limits follows from Theorem 4. The tolls can be computed based on the linear programming models presented in Fleischer et al. (2004).

# 4. Case studies

In this section, we present example results based on publicly available medium-sized TA instances (Transportation Networks for Research, 2016), which provide network data and parameters for the BPR travel time function. The instances and their characteristics are listed in Table 1. We note that units are not provided for the Berlin instances and that we

make some reasonable assumptions[2]. Because of these assumptions, results are only indicative rather than reflective of real expected travel times and fuel consumption.

**Table 1: TA network characteristics.**

|  | zones[#] | nodes | arcs | time units | distance units |
|---|---|---|---|---|---|
| **Berlin (F)*** | 23 | 224 | 523 | $sec^2$ | $metres^2$ |
| **Berlin (P)*** | 38 | 352 | 749 | $sec^2$ | $metres^2$ |
| **Berlin (M+P+F)*** | 98 | 975 | 2,184 | $sec^2$ | $metres^2$ |
| **Anaheim** | 38 | 416 | 614 | min | feet |

[#] The number of commodities (origin-destination pairs) is the square of the number of zones since the zones are the source and sink nodes.
* F: Friedrichshain, P: Prenzlauer Berg, M: Mitte.

We assume that all travel is in light vehicles and, therefore, derive an optimal speed limit of 56.495 km/h, the minimum of $c_a(v_a)$ for light vehicles with parameters from Song et al. (2013), see also Figure 2. We compute four kinds of TA solutions as shown in Table 2:

1. User equilibrium TA with respect to travel time (UE-time).
2. System-optimum TA with respect to travel time (SO-time).
3. Fuel-optimal TA as described in Section 3 (SO-fuel) with speeds $v_a^* = \min\{v_a^{opt}, v_a(f_a^*)\}$. We note that our conditions for convexity of the NLP in Section 3.2 are not fulfilled for all arcs, so that the solution SO-fuel we find may be a local optimum. The fuel savings in an optimal solution can only be larger than the results reported below.
4. User equilibrium TA assuming travel time functions $\bar{t}_a(f_a)$ have the fuel-optimal speed limits $v_a^{opt}$ strictly enforced (UE-time-limit).

**Table 2: Results for the networks from Table 1. We report on changes in both total travel time and total fuel consumption for each instance relative to the UE-time solution.**

|  |  | UE-time | SO-time | SO-fuel | UE-time-limit |
|---|---|---|---|---|---|
| **Berlin (F)** | **time** | 100.0% | 92.0% | 100.0% | 107.5% |
|  | **fuel** | 100.0% | 98.5% | 94.14% | 97.2% |
| **Berlin (P)** | **time** | 100.0% | 97.5% | 101.4% | 103.48% |
|  | **fuel** | 100.0% | 99.8% | 98.1% | 99.2% |
| **Berlin (M+P+F)** | **time** | 100.0% | 99.4% | 105.8% | 105.3% |
|  | **fuel** | 100.0% | 99.7% | 96.5% | 97.41% |
| **Anaheim** | **time** | 100.0% | 98.3% | 122.8% | 121.7% |
|  | **fuel** | 100.0% | 99.4% | 87.6% | 88.1% |

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[2] Units for the Berlin instances are not specified in the data. Since it is crucial to get correct speeds from the models to give a realistic indication of both travel time and fuel consumption, we choose time units so that free flow speeds are not completely unreasonable for inner-city travel. We confirmed on a map that it is reasonable to assume arc lengths are in metres. Assuming time in seconds and distance in metres leads to an average free flow travel speed $v_a^0$ of 116.3 km/h and a maximum $v_a^0$ of 237.6 km/h, which is very high. We finally decided to choose a somewhat unusual time unit of 2 seconds as this gives a range of speeds with an average $v_a^0$ of 58.2 km/h and a maximum $v_a^0$ of 118.8 km/h. Given there are motorways in Berlin, this seems reasonable.

For the Berlin (F) instance, the SO-time solution saves both fuel and time compared to UE-time, and the fuel-optimal solution (SO-fuel) provides further savings in total fuel consumption while the total travel time is equal to that of UE-time. In case of the Berlin (P) instance, we see less potential for improving total travel time or fuel consumption in the SO-time solution, and the SO-fuel solution is only able to decrease total fuel consumption by 1.9% while slightly increasing total travel time. For the Berlin (M+P+F) instance, SO-time is even closer to UE-time, but SO-fuel shows that a decrease of fuel consumption by 3.5% is possible while increasing total travel time by almost 6%.

Finally, the Anaheim instance shows that the SO-fuel solution can decrease total fuel consumption in the network by more than 12% while increasing total travel time to 122.8% compared to UE-time. Maximum and mean speed travelled in the UE-time solution for Anaheim is 88.5 km/h and 56.6 km/h, respectively, and it drops significantly to a maximum speed of 56.495 (the speed limit) and an average speed of 50.3 km/h when the SO-fuel solution is considered, which explains why a large decrease in fuel consumption was possible albeit at an expense of additional travel time.

The results for UE-time-limit in Table 2 demonstrate that fuel-optimal speed limits alone (without additional tolling) do push UE solutions towards the minimum total fuel consumption seen in SO-fuel. The total fuel consumption of the Anaheim instance (UE-time-limit) gets very close to that of the SO-fuel solution, whereas a larger gap remains for the Berlin (F), (P), and (M+P+F) instances. Tolls from Section 3.3 would be required to reach the SO-fuel travel pattern. The results also show that speed limits do decrease total fuel consumption but also tend to increase total travel time in the network, whereas the SO-fuel solution does not necessarily come with a great increase in travel time, as can be seen for the Berlin (F) or Berlin (P) instances.

Changes of flow comparing the SO-fuel solution to the UE-time solution are show in Figure 3 for Berlin (M+P+F). In the figure, green arcs carry higher flow in the SO-fuel solution, while red arcs carry higher flow in the UE-time solution. The magnitude of change is indicated by arc thickness. The yellow nodes are source/sink nodes, where diameter represents demand originating from this node.

**Figure 3: Berlin (M+P+F) flow differences between SO-fuel and UE-time.**

The chosen fuel-optimal speed limit is of course not practical. In reality, one would choose a speed limit with a "nice" speed value such as 50 km/h or 60 km/h, and not every road may have such a speed limit. Similar to second-best tolling approaches, an optimisation problem could be formulated where a subset of arcs are tolled or have a speed limit imposed on them. Another direction of future research is to include a realistic split of transport demand into light and heavy vehicles, which each have different fuel consumption curves (Figure 2).

# 5. Conclusions

We develop a model of fuel consumption in the context of TA. The chosen fuel consumption model is a macroscopic model that takes average arc speed as its input. Assuming that the travel time on each arc is a positive-valued, strictly increasing, continuous function of the flow on the arc and that fuel consumption follows a function by Song et al. (2013), we show that a fuel-optimal traffic pattern exists and uses travel speeds bounded from above by the speeds with minimal fuel-consumption on each arc. We show that such a fuel-optimal traffic pattern can indeed be enforced as a user equilibrium by applying both speed limits and tolls throughout the network when network users follow a generalised cost function based on a weighted sum of travel time, fuel consumption, and toll. Finally, we present results for some standard TA instances.

Throughout the paper, we discuss possible future extensions to the model such as the application of a more realistic fuel consumption model that can take into account network congestion level and acceleration / deceleration rather than just average speed. Another direction of research is the investigation of tolling or speed limits for only a subset of arcs or a discrete set of values.

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